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ON THE QUALITATIVE INVESTIGATION OF MOTIONS USING ASYMPTOTIC METHODS OF NONLINEAR MECHANICS*

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A method is proposed for qualitative estimation of oscillations defined by standard form equations, or equations with many slow and fast variables. The method makes possible the assessment of motion of the input system for all times, without having to resort to approximating exact solutions. As an example, the motion of a solid conducting body in a rapidly varying magnetic field is considered.

The most general theorems on asymptotic methods for standard form systems /1,2/ and for systems with many fast and slow variables /3/ enable us to assess the closeness of the exact and approximate solutions in a finite time interval of order $1/\varepsilon$, where ε is a small parameter. For investigating the properties of solutions of these systems in an infinite time interval we use, first, the theorems of existence of exact solutions of a specific type (e.g. quasiperiodic) of the input equation, which are obtained by the method of integral manifolds /4/ and, second, theorems on the approximation of exact solutions over an infinite interval. Among theorems of the second type the Banfi theorem /5,6/ and its extension to systems with many fast variables is often effective /6/.

However the application of such methods involves fairly rigid constraints on solutions of averaged systems, such as the requirement for uniform asymptotic stability in the case of Banfi theorem, requirement for the existence of a limit cycle in first approximation equations, etc. In particular, the case when an averaged system is in the first approximation "neutral", for instance conservative, is not included, and damping or the limit cycle are only disclosed in higher approximations. Moreover, the approximation in which, for instance, damped oscillations in an averaged system are first disclosed, does not yield an approximate solution of the original equation over an infinite time interval. This is already evident in the case of uniform exponential stability disclosed in higher approximations /7/.

Below, we present a simple method of purely qualitative analysis of motions, using asymptotic methods of nonlinear mechanics under conditions in which only the closeness of the exact solutions of the input system to the approximate solutions obtained by the method of averaging is known only in a finite time interval.

1. Qualitative comparison of exact and approximate solutions of standard form systems and of quasilinear systems with many fast variables.

We denote by $x(t)$ the unknown n-dimensional column vector that satisfies the standard form system

$$x = \varepsilon X(x, t, \varepsilon), \quad \varepsilon \geqslant 0 \tag{1.1}$$

and by $x^{(m)}(\xi^{(m)}, t, \varepsilon)$ the m-th approximation to $x(t)$ obtained by the method of averaging

$$x^{(m)} = \xi^{(m)} + \varepsilon u_1(\xi^{(m)}, t) + \ldots + \varepsilon^{m-1} u_{m-1}(\xi^{(m)}, t) \tag{1.2}$$

The equation for $\xi^{(m)}$ is of the form

$$d\xi^{(m)}/\partial t = \varepsilon \Xi_1(\xi^{(m)}) + \ldots + \varepsilon^m \Xi_m(\xi^{(m)}) = \varepsilon \Xi^{(m)}(\xi^{(m)}, \varepsilon) \tag{1.3}$$

It is assumed here that for $t \geqslant t_*$, $\varepsilon \leqslant \varepsilon_0$ and x from some domain D function X is continuous with respect to t and uniformly bounded together with the derivatives of order m with respect to x and ε, and that functions $\Xi_1, \ldots, \Xi_m$ and $u_1, \ldots, u_{m-1}$ are uniformly bounded together with the first derivatives with respect to $\xi^{(m)}$ for $\xi^{(m)} \in D$ and $\xi^{(m)} \in D$, $t \geqslant t_*$

Let us consider function $\xi_m(t, \varepsilon)$ defined by the relationship

$$x = \xi_m + \varepsilon u_1(\xi_m, t) + \ldots + \varepsilon^{m-1} u_{m-1}(\xi_m, t) \tag{1.4}$$

(subsequent reasoning remains unchanged, if, as is frequently done in the method of averaging, $\xi_m(t, \varepsilon)$ is introduced by a similar relation containing the term $O(\varepsilon^{(m)})$).

The following estimate of the closeness of functions ξ_m and $\xi^{(m)}$ is known. Let $x(t_0) \in D_\alpha$, where D_α is a domain whose α-neighborhood, $\alpha = O(\varepsilon)$, coincides with D. Using (1.4) we find that $\xi_m(t_0)$; for fairly small ε $\xi_m(t_0) \in D_\alpha$. Let the solution of Eq. (1.3) with initial condition $\xi^{(m)}(t_0) = \xi_m(t_0)$ remain in D_α within the interval $t_0 \leqslant t \leqslant t_0 + T/\varepsilon$. Then for functions $\xi^{(m)}$ and ξ_m with the indicated initial conditions and fairly small $\varepsilon \leqslant \varepsilon_*$ the relation

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$$|\xi_m(t) - \xi^{(m)}(t)| \leqslant c_m \varepsilon^m, \quad t_0 \leqslant t \leqslant t_0 + T/\varepsilon \tag{1.5}$$

where c_m and T are independent of ε, is valid.

Below, are indicated certain cases in which the properties of functions ξ_m in an infinite time interval can be established from the respective properties of functions $\xi^{(m)}$.

Let us consider some bounded domain D_1 and its δ-neighborhood D_δ, $\delta = d\varepsilon^{m-1}$, $d = \text{const} > 0$, with $D_\delta \subset D_\alpha$, and a single-valued scalar function $V(\xi, \varepsilon)$ determinate in $D_\delta, \varepsilon \leqslant \varepsilon_*$. It is assumed that $|\partial V / \partial \xi| \neq 0$ for $\xi \leqslant D_\delta$, $\varepsilon \leqslant \varepsilon_*$ and that there exist

$$V_M = \sup_{\xi \in D_\delta,\ \varepsilon \leqslant \varepsilon_*} V(\xi, \varepsilon), \quad F = \sup_{\xi \in D_\delta,\ \varepsilon \leqslant \varepsilon_*} \left| \frac{\partial V}{\partial \xi} \right|$$

Theorem 1. Let there exist on the above assumptions a function $V(\xi, \varepsilon)$ such that for any solution $\xi^{(m)}(t)$ of Eq. (1.3), which within any time interval remains of form $t^{(0)} \leqslant t \leqslant t^{(0)} + T/\varepsilon$ in D_δ, the inequality

$$V(\xi^{(m)}(t^{(0)} + T/\varepsilon)) \geqslant V(\xi^{(m)}(t^{(0)})) + \varepsilon^{m-1} W_0 \tag{1.6}$$

is satisfied for one and the same $W_0 > 0$ and all $t^{(0)} \geqslant t_*$, $\varepsilon \leqslant \varepsilon_*$ and all indicated $\xi^{(m)}(t)$.

Then there are no solutions $\xi_m(t)$ that remain in D_1 for all $t \geqslant t_*$.

Remarks. 1°. Condition (1.6) is automatically satisfied when in D_δ there exists function $V(\xi, \varepsilon)$ whose derivative by virtue of Eqs. (1.3) satisfies the relation $V^\cdot \geqslant \varepsilon^m w_0$, $w_0 = \text{const} > 0$; it is then possible to assume $W_0 = w_0 T$.

2°. It follows immediately from condition (1.6) that there are no solutions $\xi^{(m)}(t)$ that remain in D_δ for all $t \geqslant t_*$. To prove this let us assume that $\xi^{(m)}(t_0) \in D_\delta$ and consider the time interval $t_0 \leqslant t \leqslant t_0 + kT/\varepsilon$, where k is an integer. At the end of this time interval function V in any solution remaining in D_δ acquires an increment $V(\xi^{(m)}(t_0 + kT/\varepsilon)) - V(\xi^{(m)}(t_0)) \geqslant k\varepsilon^{m-1} W_0$. As the result, the value of function $V(\xi, \varepsilon)$ in solution $\xi^{(m)}(t)$ will exceed V_M for a fairly large k, which is impossible.

Proof. Let us compare the sequence of "approximate" solutions $\xi_j^{(m)}(t)$, $j = 0, 1, \ldots$, defined by conditions

$$\xi_0^{(m)}(t_0) = \xi_m(t_0), \quad \xi_1^{(m)}(t_0 + T/\varepsilon) = \xi_m(t_0 + T/\varepsilon), \quad \ldots, \quad \xi_j^{(m)}(t_0 + jT/\varepsilon) = \xi_m(t_0 + jT/\varepsilon)$$

with the "exact" solution $\xi_m(t)$, $\xi_m(t_0) \in D_1$, and show that it is impossible for $\xi_j^{(m)}(t) \in D_\delta$ to exist in the intervals $t_0 + jT/\varepsilon \leqslant t \leqslant t_0 + (j+1)T/\varepsilon$ for arbitrarily large j. Let $\xi_j^{(m)}(t) \in D_\delta$, $t_0 + jT/\varepsilon \leqslant t \leqslant t_0 + (j+1)T/\varepsilon$ for all j. Then $\xi_m(t_0 + jT/\varepsilon) \in D_\delta$ for all j. In conformity with the theorem function $V(\xi, \varepsilon)$ in solution $\xi_0^{(m)}(t)$ acquires for any $t_0 \leqslant t \leqslant t_0 + T/\varepsilon$ the increment

$$V(\xi_0^{(m)}(t_0 + T/\varepsilon)) - V(\xi_0^{(m)}(t_0)) \geqslant \varepsilon^{m-1} W_0 \tag{1.7}$$

Let us consider the quantity $V(\xi_m(t_0 + T/\varepsilon))$. In conformity with (1.5)

$$|V(\xi_m(t_0 + T/\varepsilon)) - V(\xi_0^{(m)}(t_0 + T/\varepsilon))| \leqslant F\,|\xi_m(t_0 + T/\varepsilon) - \xi_0^{(m)}(t_0 + T/\varepsilon)| \leqslant F c_m \varepsilon^m \tag{1.8}$$

from which follows

$$V(\xi_m(t_0 + T/\varepsilon)) \geqslant V(\xi_0^{(m)}(t_0 + T/\varepsilon)) - \varepsilon^m F c_m \tag{1.9}$$

Taking into account that $V(\xi_0^{(m)}(t_0)) = V(\xi_m(t_0))$, and using the notation $W = W_0(1 - \varepsilon_* F c_m / W_0)$, we obtain from (1.7) and (1.9) the estimate

$$V(\xi_m(t_0 + T/\varepsilon)) - V(\xi_m(t_0)) \geqslant \varepsilon^{m-1} W \tag{1.10}$$

In the same manner (comparing with the increment $V(\xi_1^{(m)}(t))$ in the interval $t_0 + T/\varepsilon \leqslant t \leqslant t_0 + 2T/\varepsilon$) we can estimate $V(\xi_m(t_0 + 2T/\varepsilon))$. Similarly to (1.10) we obtain

$$V(\xi_m(t_0 + 2T/\varepsilon)) - V(\xi_m(t_0 + T/\varepsilon)) \geqslant \varepsilon^{m-1} W$$

which with (1.10) implies that

$$V(\xi_m(t_0 + 2T/\varepsilon)) - V(\xi_m(t_0)) \geqslant 2\varepsilon^{m-1} W \tag{1.11}$$

Using $j+1$ functions $\xi_0^{(m)}, \ldots, \xi_j^{(m)}$ we obtain

$$V(\xi_m(t_0 + (j+1)T/\varepsilon)) - V(\xi_m(t_0)) \geqslant (j+1)\varepsilon^{m-1} W \tag{1.12}$$

which shows that for fairly large j the quantity $V(\xi_m(t_0 + (j+1)T/\varepsilon))$ exceeds V_M, which is impossible.

Consequently there exist such k and t_1, $t_0 + kT/\varepsilon \leqslant t_1 \leqslant t_0 + (k+1)T/\varepsilon$ that $\xi_k^{(m)}(t_1) \,\overline{\in}\, D_\delta$. Then by virtue of (1.5) and of the relation $\delta = d\varepsilon^{m-1}$ we obtain $\xi_m(t_1) \,\overline{\in}\, D_1$.

We prove in exactly the same way that $\xi_m(t)$ is out not only of D_1 but, also, from any

$\theta\delta$ -neighborhood of D_1, $\theta < 1$, θ is independent of ε.

It is useful to evaluate the time Δt in which solution $\xi_m(t)$ will necessarily leave domain D_1

$$\Delta t \leqslant \frac{V_M}{\varepsilon^m W} = \frac{T_1}{\varepsilon^m}$$

Theorem 2. Let function V satisfy the conditions of Theorem 1 and, furthermore, the derivative $V^{\cdot}$ calculated with the use of Eqs. (1.3) be nonnegative throughout D_δ. Let domain D_1 be bounded by surfaces $V = C_1$ and $V = C_2$, $C_2 > C_1$, and all surfaces of the set $V = C$ be closed. Then any solution $\xi_m(t)$, $\xi_m(t_0) \in D_1$ intersects surface $V = C_2$ after some time, and leaves forever domain D_1.

Remark. We denote by $S(C)$ a surface of the form $V = C$. From the defined above properties of the derivative $\partial V/\partial\xi$ follows that the set $S(C)$ has no singularities in D_δ. Since the surfaces are closed, they envelope each other. Let us suppose that surface $S(C')$ envelopes $S(C'')$ for any $C' < C''$; this enables us to use the expressions "outside $S(C_1)$", "inside $S(C_2)$", etc. The case when $S(C'')$ envelops $S(C')$ is similarly considered.

Proof. Let us, first, show that $\xi_m(t)$ will necessarily appear inside $S(C_2)$. The "approximate" solution $\xi_0^{(m)}(t)$ (see above) can leave D_δ only by intersecting $S(C_2)$. If $\xi_0^{(m)}(t)$ leaves D_δ in the interval $t_0 \leqslant t \leqslant t_0 + T/\varepsilon$, then $\xi_m(t)$ leaves D_1 in the same interval through $S(C_2)$ (if $\xi_m(t_0)$ lies close to the boundary $S(C_1)$, then $\xi_m(t)$ may "on the way" out of D_1 pass through that boundary, (but this is immaterial).

Let $\xi_0^{(m)}(t) \in D_\delta$ when $t_0 \leqslant t \leqslant t_0 + T/\varepsilon$. Point $\xi_0^{(m)}(t_0 + T/\varepsilon)$ lies inside $S(C_1)$ at distance ρ from it. The estimate for ρ (similarly to (1.8)) is $\rho \geqslant \varepsilon^{m-1}W_0/F$. Point $\xi_m(t_0 + T/\varepsilon)$ by virtue of (1.5) is also inside $S(C_1)$ at distance $\rho_1 = O(\rho)$ from it. Examination of function $\xi_1^{(m)}(t)$ shows that point $\xi_m(t_0 + 2T/\varepsilon)$ lies inside $S(C_1)$ at distance $\rho_2 \geqslant \rho_1 + \varepsilon^{m-1}W/F$ from the boundary. In the second interval, point $\xi_m(t)$ cannot appear outside $S(C_1)$. Then we have $\rho_3 \geqslant \rho_1 + 2\varepsilon^{m-1}W/F$, etc. Consequently, point $\xi_m(t)$ which in conformity with Theorem 1 must leave D_1 does so through $S(C_2)$.

Let now $\xi_m(t_1)$ lie inside $S(C_2)$, and let a certain number of solutions $\xi_j^{(m)}(t)$, $\xi_0^{(m)}(t_1) = \xi_m(t_1)$, etc. remain in D_δ. The solution $\xi_m(t)$ cannot get beyond $S(C_2)$ already in the second interval $t_1 + T/\varepsilon \leqslant t \leqslant t_1 + 2T/\varepsilon$, in the same way as previously solution $\xi_m(t)$ could not get beyond $S(C_1)$ in the corresponding interval. We are left with the case when some solution $\xi_k^{(m)}(t)$ leaves D_δ. We denote by C_M the maximum value of C for which $S(C)$ lies completely in D_δ; $C_M - C_2 = O(\varepsilon^{m-1})$. In the interval $t_1 + kT/\varepsilon \leqslant t \leqslant t_1 + (k+1)T/\varepsilon$ point $\xi_m(t)$ may lie outside $S(C_M)$ only at the distance $O(\varepsilon^m)$ from the boundary. It is now obvious that independently of whether $\xi_{k+1}^{(m)}(t)$ gets or does not get beyond D_δ, $\xi_m(t)$ does not reach D_1 also in the second interval.

The following theorem follows from Theorem 2.

Theorem 3. Let the surfaces $S(C)$ contract to a point as $C \to C_*$. Let also for any arbitrarily small $\eta > 0$ exists a $\varepsilon(\eta)$ such that Theorem 2 is satisfied for $\varepsilon \leqslant \varepsilon(\eta) \leqslant \varepsilon_*$ and $C_2 - C_* = \eta$. Then for fairly small ε the solution $\xi_m(t)$ beginning at some $t = t(\eta)$ remains for ever in any arbitrarily small η-neighborhood of point $V = C_*$.

In accordance with (1.4) solutions $x(t)$ of the input system (1.1) under conditions of Theorem 3 and fairly large t remain in the $(\eta + O(\varepsilon))$-neighborhood of point $V = C_*$. Let point $V = C_*$ correspond to the equilibrium position of some mechanical system. Then the oscillations defined by the input system (1.1) qualitatively represent the superposition of form (1.4) of a slow evolutionary motion approaching the "nearly" equilibrium position and small rapid vibrations. For fairly considerable t the motion reduces to small rapid oscillations about the mean position which may be slowly wandering in a small region. Such oscillations differ only little from quasistatic.

When surface $S(C'')$ envelops $S(C')$ and $C'' > C'$, we have from Theorem 2 that solution $\xi_m(t)$ moves for ever beyond surface $S(C_2)$. If Theorem 2 holds for fairly large, or even arbitrarily large C_2, then $x(t)$ defines the superposition of small vibrations on the "departing" motion.

After some evident alterations in Theorems 1—3 it is possible to assume the existence not of increasing but of decreasing functions V; for instance, it is possible to specify in Theorem 1 the condition

$$V(\xi^m(t^{(0)} + T/\varepsilon)) - V(\xi^{(m)}(t^{(0)})) \leqslant -\varepsilon^{m-1}W_0$$

assuming that $\inf V$ exists in D_δ. The use of such functions is particularly convenient in the example in Sect. 2.

It is possible to determine function V when the first or several lower approximations of Eq. (1.3) admit the first integral $V(\xi, \varepsilon) = \text{const}$, while in the subsequent approximation that integral vanishes, and the sign of $V^{\cdot}$ can be established. In this case we obviously have $V^{\cdot} = O(\varepsilon^m)$, where m is the ordinal number of the approximation in which the integral disappears for the first time. The simplet is the most often occuring case when $V = \text{const}$ is the energy integral and the relation $V^{\cdot} = 0$ is violated because of dissipation which is disclosed in higher approximations.

The above theorems can be extended to systems other than the standard form system, if it is possible to prove the closeness of the exact and approximate solutions in the interval T / ε, and the derivation of the approximate solution reduces to the integration of an autonomous system. Such are, for example, quasilinear systems with many fast variables.

$$x^{\cdot} = \varepsilon X(x, y, t, \varepsilon), \quad y^{\cdot} = A(x) y + f(x, t) \tag{1.13}$$

where x and y are the sought n- and l-dimensional column vectors, and matrix $A(x)$ is such that its eigenvalues $\lambda_1(x), \ldots, \lambda_l(x)$ satisfy the condition $\operatorname{Re} \lambda_i \leqslant -\mu < 0, \ \mu = \text{const}$.

For system (1.13) we have /7,8/

$$|\xi_m - \xi^{(m)}| \leqslant c_m \varepsilon^m, \ |x - x_m| \leqslant c^{(m)} \varepsilon^m, \ |x - x_m^{(m)}| \leqslant c_m^{(m)} \varepsilon^m, \ |y - y^{(m-1)}(\xi^{(m)}, t, \varepsilon)| \leqslant b_m \varepsilon^m \tag{1.14}$$

where

$$y^{(m-1)}(\xi_m, t, \varepsilon) = \varphi^{(m-1)}(x(\xi_m, t, \varepsilon), t, \varepsilon) = \sum_{i=0}^{m-1} \varepsilon^i \varphi_i(x, t, \varepsilon) \tag{1.15}$$

and functions $\varphi_0, \varphi_1, \ldots, \varphi_{m-1}$ are determined by the linear equations

$$\varphi_0^{\cdot} = A\varphi_0 + f, \quad \varphi_1^{\cdot} = A\varphi_1 - \frac{\partial \varphi_0}{\partial x} X(x, \varphi_0, t, 0) \tag{1.16}$$

etc. which are integrated with $x = \text{const}$ and initial conditions $\varphi_0(x(t_0), t_0) = y(t_0)$ and $\varphi_i(x(t_0), t_0) = 0$ (for details see /7,8/). In (1.14) x_m denotes the solution of the standard form system

$$x_m^{\cdot} = \varepsilon X(x_m, \varphi^{(m-1)}(x_m, t, \varepsilon), t, \varepsilon) \tag{1.17}$$

$x_m^{(m)}$ is the m-th approximation to x_m of form (1.2), and ξ_m and $\xi^{(m)}$ are introduced in system (1.17) as in system (1.1). An autonomous system of form (1.3) is obtained for $\xi^{(m)}$; then $\xi^{(m)}$ and ξ_m can be qualitatively compared using the theorem proved above. After this it is possible to establish the respective properties of the sought functions $x(t)$ and $y(t)$.

2. Example. Motion of a conducting solid body in a high-frequency magnetic field. The motion of a conducting solid body in a high-frequency magnetic field and the Foucault currents in the body are defined by the equations /9/

$$q^{\cdot} = \varepsilon A^{-1}(q) p, \quad p^{\cdot} = \varepsilon\left(-\frac{1}{2} p^T \frac{\partial A^{-1}}{\partial q} p + J\left(\frac{\partial L_e}{\partial q}\right)^T i\right) + \varepsilon^2 Q(q, p), \quad Li^{\cdot} + Ri = -(L_e J)^{\cdot} \tag{2.1}$$

where q and p are n-dimensional column vectors of dimensionless mechanical coordinates and momenta, $A(q)$ is the matrix of inertia coefficients, $\varepsilon^2 Q$ is the vector of generalized forces, i is the infinite-dimensional vector of dimensionless Foucault currents in the body, L and R are infinite-dimensional of induction coefficients and reciprocal resistance of nominal contours in which the body is divided /10/, $J(t)$ is the specified current in the contour which generates the external field, L_e is the vector of coefficients of reciprocal induction of the contour with current J and the contour of Foucauld currents in the body, and t is the dimensionless time. The symbol $p^T(\partial A^{-1}/\partial q) p$ denotes a vector whose j-th component is $p^T(\partial A^{-1}/\partial q_j) p$. For equations defining electromechanical systems in discrete form see /10/; Chapt.VII.

In technical problems, such as that of orienting components by variable magnetic field, the "external" generalized forces are represented by friction, for instance viscous friction.

System (2.1) comprises fast i and slow q and p variables and, being a particular case of (1.13) may be analyzed by the above method (a more complicated and more general method of V. M. Volosov was used in /5/).

Let us consider the second approximation. After the elimination of fast variables, as indicated above, we obtain a standard form system in q_2 and p_2. Its approximate solution is of the form

$$q_2^{(2)} = \xi^{(2)} + \varepsilon u_1(\xi^{(2)}, \eta^{(2)}, t), \quad p_2^{(2)} = \eta^{(2)} + \varepsilon v_1(\xi^{(2)}, \eta^{(2)}, t)$$

We specify functions u_1 and v_1 so that their time averaged values $\langle u_1 \rangle$ and $\langle v_1 \rangle$ vanish, eliminating by this the arbitrariness of their selection. The equations for $\xi^{(2)}$ and $\eta^{(2)}$ then assume the form /9/

$$\xi^{(2)\cdot} = \varepsilon A^{-1}(\xi^{(2)}) \eta^{(2)}, \quad \eta^{(2)\cdot} = \varepsilon\left(-\frac{1}{2} \eta^{(2)T} \frac{\partial A^{-1}}{\partial \xi^{(2)}} \eta^{(2)} - \frac{\partial \Lambda}{\partial \xi^{(2)}} - B(\xi^{(2)}) \xi^{(2)\cdot}\right) + \varepsilon^2 Q_2 \tag{2.2}$$

where $\Lambda = \langle W(i_0) \rangle$, $W = {}^1/_2\, i^T L i$ is the energy of the magnetic field of Foucauld currents in the body, and $i_0(\xi^{(2)})$ are the Foucauld currents obtained in the generating approximation, i.e. from the equation

$$Li_0^{\cdot} + Ri_0 = -L_c(\xi^{(2)}) J^{\cdot} \tag{2.3}$$

in which $\xi^{(2)}$ is assumed to be a constant quantity. In other words, the mean value of the magnetic field energy obtained in the generating approximation is used in (2.3). The term $(-\varepsilon B \xi^{(2)\cdot})$ is a second order quantity, since $\xi^{(2)\cdot} = O(\varepsilon)$. Matrix B is symmetric /9/, hence that term defines either dissipative or "swinging" forces. The effect of the magnetic field on the slow motions, thus, results in the appearance in the first approximation of potential forces and in the second of dissipative forces.

Let us consider system (2.2) in the first approximation, i.e. without terms $O(\varepsilon^2)$. Such system is conservative with the Hamiltonian

$$H = \frac{\varepsilon}{2} \eta^{(2)T} A^{-1}(\xi^{(2)}) \eta^{(2)} + \varepsilon\Lambda$$

Hence we take function V of the form $V = H / \varepsilon$.

Let the first approximation function have a stable equilibrium position which is surrounded by closed surfaces $V = C$, $C =$ const. Let in the equilibrium position $\Lambda = 0$. We assume domain D_1 to be bounded by two surfaces $S(C_1)$ and $S(C_2)$, $C_1 > C_2$, C_1, $C_2 = O(1)$, the first of which envelopes the second. By virtue of Eqs. (2.2) the derivative $V^{\cdot}$ is

$$V^{\cdot} = -\varepsilon^2 (A^{-1}\eta^{(2)})^T (B + G)(A^{-1}\eta^{(2)}) \tag{2.4}$$

where G is the matrix of coefficients of external viscous friction forces. Let us consider the case when matrix B is positive definite. It can be shown that the inequality

$$V(\xi^{(2)}(t^{(0)} + T/\varepsilon)) - V(\xi^{(2)}(t^{(0)})) \leqslant -\varepsilon^{m-1} W_0 \tag{2.5}$$

Indeed, if $|\eta^{(2)}(t^{(0)})| = O(1)$, then by virtue of (2.4) the increment of function V in solution $\xi^{(2)}(t), \eta^{(2)}(t)$ over any Δt time interval is no greater than $(-\varepsilon^2 \text{ const } \Delta t)$. If, however, $|\eta^{(2)}(t^{(0)})|$ is small, then over a time of order

$$\Delta t = \text{const}\left(\varepsilon \inf_{D_1} \left| \frac{\partial \Lambda}{\partial \xi^{(2)}} \right|\right)^{-1}$$

we have, in conformity with the second of Eqs. (2.2), $|\eta^{(2)}| = O(1)$. Selecting $T = \varepsilon k \Delta t, k > 1$, which affects only the constant c_m in estimate $|\xi_m - \xi^{(m)}| \leqslant c_m \varepsilon^m$, we obtain the required inequality.

Inequality (2.5) and formulas (2.4) are satisfied for any fairly large C_2. This makes possible the application of Theorem 3. As the result, we find that the phase trajectory $\xi^{(2)}(t)$, $\eta^{(2)}(t)$ appearing in domain D_1 of the form described above, for fairly small ε ultimately reaches some small η-neighborhood of the equilibrium position, and remains there. To this correspond input system oscillations that are qualitatively similar to damped oscillations approaching the quasistatic mode (further investigation is, however, required for proving that solutions of the input system approach quasistatic state). Similarly, when the matrix $B + G$ of total friction is negative definite, we have "increasingly swinging" oscillations.

But it is necessary to prove in addition that oscillations under initial conditions close to the equilibrium position are so enhanced.

Since matrix B depends on $\xi^{(2)}$, a case is possible in which the total nonpotential forces $\varepsilon^2 Q_2 - \varepsilon B \xi^{(2)\cdot}$ enhance oscillations near the equilibrium position, and are dissipative away from it. In such cases a limit cycle is possible. It can be similarly proved, at least in the case of a single mechanical degree of freedom, that $\xi_2(t)$ reaches a small neighborhood of that cycle. In the case of periodic function $J(t)$ the oscillations for considerable t are qualitatively similar to quasistatic, and the system motions resemble either damped or increasing oscillations approaching the quasiperiodic mode. We would, however, point out that the existence of quasiperiodic solutions in cases when a limit cycle is disclosed in higher approximations, has apparently not been proved.

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